

MSOL Restricted Contractibility to Planar Graphs[☆]

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Abstract

We study the computational complexity of graph planarization via edge contraction. The problem `CONTRACT` asks whether there exists a set S of at most k edges that when contracted produces a planar graph. We work with a more general problem called `P-RESTRICTEDCONTRACT` in which S , in addition, is required to satisfy a fixed MSOL formula $P(S, G)$. We give an FPT algorithm in time $\mathcal{O}(n^2 f(k))$ which solves `P-RESTRICTEDCONTRACT`, where n is number of vertices of the graph and $P(S, G)$ is (i) inclusion-closed and (ii) inert contraction-closed (where inert edges are the edges non-incident to any inclusion-minimal solution S).

As a specific example, we can solve the ℓ -subgraph contractibility problem in which the edges of the set S are required to form disjoint connected subgraphs of size at most ℓ . This problem can be solved in time $\mathcal{O}(n^2 f'(k, \ell))$ using the general algorithm. We also show that for $\ell \geq 2$ the problem is NP-complete.

Keywords: planar graph, contraction, MSOL formula, FPT algorithm

1. Introduction

Graph visualization techniques are thoroughly studied. In many applications visual understanding of the graph under consideration is important or required. It is commonly accepted that edge crossings make a plane drawing of a graph less clear, and thus the goal is to avoid them, or reduce their number. It is now well-known that one can decide fast whether crossings can be avoided at all, as planarity testing is linear time decidable [2], while determining the minimum number of crossings needed to draw a non-planar graph is NP-hard [3]. Several variants of planar visualization of graphs have been considered and explored, including simultaneous embeddings, book-embeddings, embeddings on surfaces of higher genus, etc.

Marx and Schlotter [4] considered planarization of a graph by removing its vertices while Kawarabayashi and Reed [5] considered removing its edges. Another possible way to planarize a graph is by contracting some of its edges. If the number of contracted edges is not limited, every connected graph can trivially be contracted into a single vertex, and thus becomes planar. A graph is *k-contractible* if the number of contracted edges is limited by a number k . If k is a part of the input, testing *k-contractibility* is NP-complete [6]. Polynomial-time algorithms are known if one asks about contraction to a particular fixed planar graph (so called *H-contractibility*); for nice overviews see [7, 8]. In this paper, we present a fixed-parameter tractable algorithm for contractibility to planar graphs.

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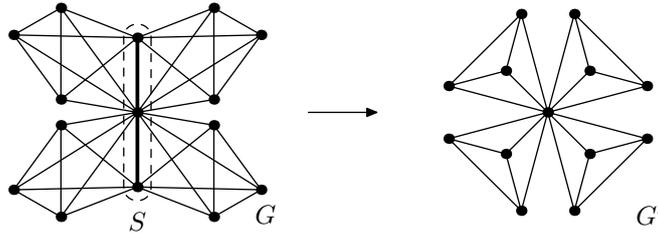


Figure 1: An example of a 2-contractible graph.

Definitions and Notation. In this paper all graphs are simple, i.e., no multiple edges and no loops. For a graph G , we denote by $V(G)$ its vertices and by $E(G)$ its edges (or simply V and E , when the graph is clear from the context). By $G \setminus H$, we denote the subgraph of G induced by $V(G) \setminus V(H)$. We denote by $G \circ e$ the graph obtained by contracting an edge e in G . For a set of edges S , we denote a graph created from G by contracting all edges of S by $G \circ S$. We call $S \subseteq E$ a *planarizing set* of G if $G \circ S$ is a planar graph; see Fig. 1. We say that G is *k-contractible* if there exists a planarizing set S of size at most k .

MSOL formulas for graphs are logic formulas which contain predicates of equality, incidence and containment, logic operators and quantifiers for vertices, edges, sets of vertices and edges; see [9]. For instance, 3-colorability can be expressed by MSOL as existence of three sets V_1, V_2 and V_3 of vertices, such that each vertex belongs to exactly one V_i , and there are no edges with both endpoints in one V_i .

Restricted Contractibility. We address the following more general problem. We want to find a planarizing set S of size at most k that satisfies an additional restriction: a monadic second-order logic (MSOL) formula $P(S, G)$ fixed for the problem.¹

Problem:	P -RESTRICTEDCONTRACT
Input:	An undirected graph G and an integer k .
Output:	Is there a planarizing set $S \subseteq E(G)$ of size at most k satisfying $P(S, G)$ that when contracted produces a planar graph?

We want to construct an FPT algorithm for this problem with respect to the parameter k . This is not possible for every MSOL formula $P(S, G)$. For some formulas, the problem is already NP-hard even for $k = 0$. For instance, let $P(S, G)$ be the formula: “For $S = \emptyset$, is $G \circ S$ a 3-colorable graph?” Then the problem P -RESTRICTEDCONTRACT is equivalent to testing 3-colorability of planar graphs which is known to be NP-complete [10].

In this paper, we describe an FPT algorithm which works as follows. Either the graph is simple (of small tree-width) and the problem can be solved in a brute-force way. Or we find a small part of the graph which we can prove to be far from any inclusion-minimal planarizing set. We modify the graph by contracting this small part, and repeat the process. Therefore, we need to restrict ourselves to MSOL formulas for which satisfiability is not changed by this modification.

An MSOL formula P is *inclusion-closed* if for every S satisfying P also every $S' \subseteq S$ satisfies P . This property is necessary since the algorithm looks for inclusion-minimal planarizing sets. A set B of edges of G is called *inert* if it is not incident with any edge of any inclusion-minimal planarizing set S ; see Fig. 2a for an example. A formula P is called *inert contraction-closed* if the following holds for every inclusion-minimal planarizing set S and every inert set B

$$P(S, G) \iff P(S, G \circ B)$$

Therefore the modification by contraction of inert edges does not change solvability of the problem.

¹More precisely, we have different formulas $P_k(e_1, \dots, e_k)$ for each k where $S = \{e_1, \dots, e_k\}$. So the length of the formula may depend on k .

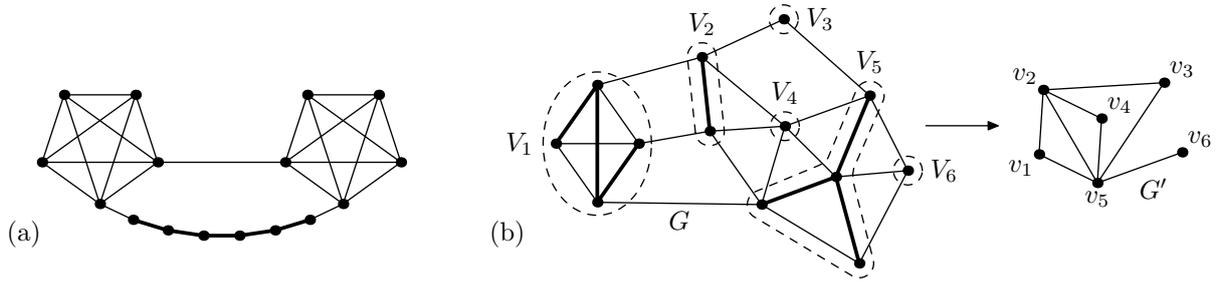


Figure 2: (a) Every inclusion-minimal planarizing set S contains one edge in each K_5 . Therefore, inert sets are subsets of the highlighted edges. (b) The connected components of a planarizing sets S act as clusters. After contracting S , each cluster corresponds to one vertex in G' . Two vertices in G' are adjacent if and only if there exists an edge in G between the corresponding clusters.

Theorem 1.1. *For every inclusion-closed and inert contraction-closed MSOL formula P , the problem P -RESTRICTEDCONTRACT is solvable in time $\mathcal{O}(n^2 f(k))$ where n is the number of vertices of G and f is a computable function.*

Our algorithm uses an approach developed by Grohe [11] which shows that there is a quadratic-time FPT algorithm for crossing number. The most significant difference is the proof of Lemma 2.3. We cannot use the same approach as that of [11] because k -contractible graphs do not have bounded genus [12] which is essential in [11]. Further, our approach in the proof of Lemma 2.3 can be modified for the crossing number which simplifies the argument of Grohe [11]; see Section 4.

For a trivial formula P that is true for every set of edges, we get the k -contractibility problem considered above. We note that k -contractibility was independently proved to be solvable in time $\mathcal{O}(n^{2+\varepsilon} f(k))$ for every $\varepsilon > 0$ in a recent paper of Golovach et al. [12]. The algorithm described here uses similar techniques but has a better time complexity.

ℓ -subgraph Contractibility. For different formulas P , we get problems different from k -contractibility having new specific properties. As one particular example, we work with a problem which we call ℓ -subgraph contractibility. A graph is called ℓ -subgraph contractible if and only if there exists a planarizing set S such that its edges form disjoint connected subgraphs with at most ℓ vertices. For instance, for $\ell = 2$ the planarizing set S is required to be a matching, see Fig. 3.

Problem:	ℓ -SUBCONTRACT
Input:	An undirected graph G and an integer k .
Output:	Is G ℓ -subgraph contractible by a set S having at most k edges?

Contraction of a set S can be interpreted as graph clustering, see Fig. 2b. We want to find clusters such that the resulting cluster graph is planar. For ℓ -subgraph contractibility, every cluster has to be of size at

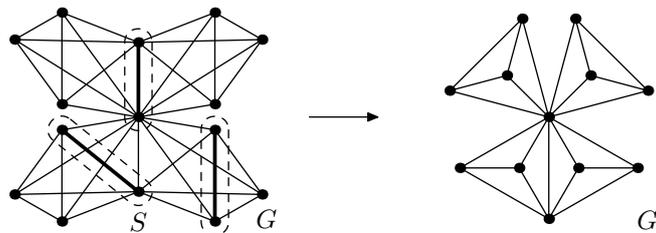


Figure 3: For $\ell = 2$, when we require that a planarizing set S is a matching, a smallest planarizing set contains three edges.

most ℓ . In comparison to k -contractibility, the contracted edges have to be more equally distributed in G , and thus the contractions do not change the graph too much.

From a graph drawing perspective this approach offers a drawing such that all crossings happen in disjoint areas nearby the clusters and the rest of the meta-drawing is crossing-free. Such a meta-drawing resembles well the original graph and can be well grasped by a glance from the distance. The local crossings get inspected by taking a magnifying glass for particular clusters.

If $\ell = 1$, the problem is solvable in linear time as it becomes just planarity testing. For $\ell \geq 2$, we prove:

Proposition 1.2. *For $\ell \geq 2$, the problem ℓ -SUBCONTRACT is NP-complete.*

Since ℓ -subgraph contractibility can be expressed using MSOL formulas, we get the following corollary of Theorem 1.1.

Corollary 1.3. *For every fixed ℓ , the problem ℓ -SUBCONTRACT can be solved in time $\mathcal{O}(n^2 f_\ell'(k))$ where n is the number of vertices and f_ℓ' is a computable function.*

Paper Layout. In Section 2, we describe our FPT algorithm for the P -RESTRICTEDCONTRACT problem. In Section 3, we deal with the ℓ -SUBCONTRACT problem. Last, in Section 4, we show how to simplify the proof of Grohe [11].

2. Restricted Contractibility is Fixed-Parameter Tractable

Let P be a fixed inclusion-closed and inert contraction-closed MSOL formula. In this section, we show that the problem P -RESTRICTEDCONTRACT is fixed-parameter tractable with respect to the parameter k . Namely, we describe an algorithm which solves P -RESTRICTEDCONTRACT in time $\mathcal{O}(n^2 \cdot f(k))$ for some function f .

The basic structure of our algorithm is based on the following idea invented by Grohe [11]. If the graph has a small tree-width, we solve the problem by Courcelle's Theorem [13]. If the tree-width is large, we find an embedded large hexagonal grid and produce a smaller graph to which we apply the procedure recursively.

2.1. Definitions

We first introduce notation similar to that of Grohe's in [11].

Topological Embeddings. A topological embedding $h : G \hookrightarrow H$ of G into H consists of two mappings: $h_V : V(G) \rightarrow V(H)$ and $h_E : E(G) \rightarrow P(H)$, where $P(H)$ denotes set of all paths in H . These mappings must satisfy the following properties:

- The mapping h_V is injective, distinct vertices of G are mapped to distinct vertices of H .
- For distinct edges e and f of G , the paths $h_E(e)$ and $h_E(f)$ are distinct, do not share internal vertices and share possibly at most one endpoint.
- If $e = uv$ is an edge of G then $h_V(u)$ and $h_V(v)$ are the endpoints of the path $h_E(e)$. If w is a vertex of G different from u and v then path $h_E(e)$ does not contain the vertex $h_V(w)$.

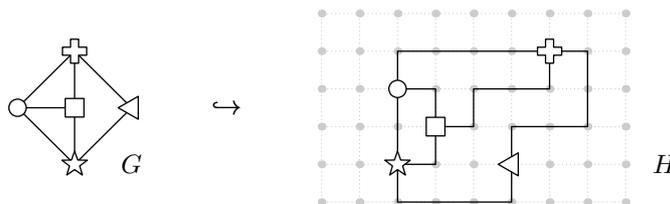


Figure 4: An example of a topological embedding $G \hookrightarrow H$.

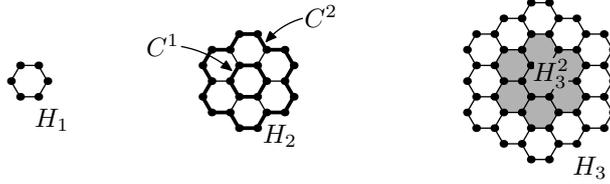


Figure 5: Hexagonal grids H_1 , H_2 and H_3 . Inside H_2 , the principal cycles C^1 and C^2 are depicted in bold. Inside H_3 , the principal subgrid H_3^2 is shown.

For an example, see Fig. 4.

It is useful to notice that there exists a topological embedding $h : G \hookrightarrow H$, if there exists a subdivision of G which is a subgraph of H . For a subgraph $G' \subseteq G$, denote by $h \upharpoonright G'$ the restriction of h to G' . For simplicity, we use the term *embeddings* instead of topological embeddings.

Hexagonal grid. We define recursively the hexagonal grid H_r of radius r (see Fig. 5). The graph H_1 is a hexagon (the cycle of length six). The graph H_{r+1} is obtained from H_r by adding $6r$ hexagonal faces around H_r as indicated in Fig. 5.

The nested *principal cycles* C^1, \dots, C^r are called the boundary cycles of H_1, \dots, H_r . From the inductive construction of H_r , H_k is obtained from H_{k-1} by adding C^k and connecting it to C^{k-1} . A *principal subgrid* H_r^s where $s \leq r$ denotes the subgraph of H_r isomorphic to H_s and bounded by the principal cycle C^s of H_r .

Flat Topological Embeddings. Let H be a subgraph of a graph G . An H -component C of G is

- either a connected component of $G \setminus H$ together with the edges connecting C to H and its incident vertices, or
- an edge $e = uv$ and the incident vertices u and v such that $u, v \in V(H)$ and $e \notin E(H)$.

The endpoints of edges of C contained in H are called the *vertices of attachment* of C . Figure 6a illustrates the notion of H -components.

Let G be a graph and let $h : H_r \hookrightarrow G$ be an embedding of a hexagonal grid in G . A vertex $v \in h(H_r)$ is called *inner* if $v \in h(H_r) \setminus h(C^r)$. An $h(H_r)$ -component C is called *proper* if C has at least one vertex of attachment in $h(H_r) \setminus h(C^r)$, namely, the component is attached to an inner vertex of the grid. Let $h_+(H_r)$ denote the union of $h(H_r)$ with all proper $h(H_r)$ -components. Notice that the proper $h(H_r)$ -components may be obstructions to the planarity of $h_+(H_r)$. The embedding h is called a *flat embedding* if $h_+(H_r)$ is a planar graph. For an example, see Fig. 6b.

Tree-width. For a graph G , its tree-width is an integer k which describes how “similar” is G to a tree [14]. For our purposes, we use tree-width as a black box in our algorithm. The following two properties of tree-width are crucial.

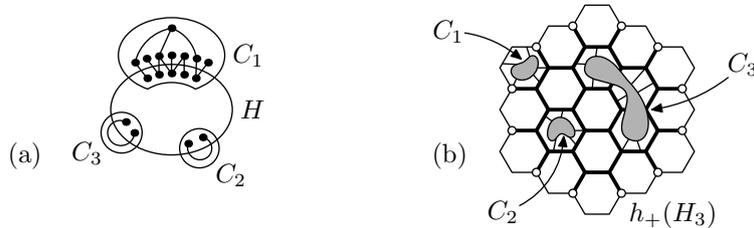


Figure 6: (a) H -components C_1 , C_2 and C_3 of G . (b) A subgraph $h_+(H_3)$ consisting of $h(H_3)$ and three proper components C_1 , C_2 and C_3 having attachments to inner vertices of $h(H_3)$ (highlighted in bold). The embedding h is not flat since C_3 obstructs planarity.

Theorem 2.1 (Robertson and Seymour [15], Boadlaender [16], and Perkovič and Reed [17]). For every $s \geq 1$, there is $t \geq 1$ and a linear-time algorithm that, given a graph G , either (correctly) recognizes that the tree-width of G is at most t or returns an embedding $h : H_s \hookrightarrow G$.

Theorem 2.2 (Courcelle [13]). For every graph G of tree-width at most t and every MSOL formula φ , there exists an algorithm that decides the formula φ on G in time $\mathcal{O}(n \cdot g(t, |\varphi|))$, where n is the number of vertices of G .

2.2. The algorithm

Overview. The general outline of the algorithm is as follows. It proceeds in two phases. The first phase deals with graphs of large tree-width and repeatedly modifies G to produce a graph of small tree-width. In addition, we keep a set $F \subseteq E$ of forbidden edges for contractions. Initially, F is empty and during the modification some edges are added. The second phase reduces P -RESTRICTEDCONTRACT to solving an MSOL formula which is done by Courcelle's Theorem 2.2.

Phase I. We first prove the following lemma, which states that in an embedded large hexagonal grid H_s into G , we either find a flat hexagonal grid H_r (smaller than H_s), or else G is not k -contractible. This lemma represents the most significant difference from the paper of Grohe [11], as illustrated in Section 4.

Lemma 2.3. Let G be a k -contractible graph. For every $r \geq 1$, there exists $s \geq 1$ such that for every embedding $h : H_s \hookrightarrow G$ there is some subgrid $H_r \subseteq H_s$ such that $h \upharpoonright H_r$ is a flat embedding.

PROOF. For given r and k , we fix s and t large enough as follows: We choose $s \approx 2kt$ so that H_s contains $2k + 1$ disjoint subgrids $H_{t_1}, \dots, H_{t_{2k+1}}$ of radius t . Let H'_{t_i} , formerly denoted by $H_{t_i}^{t-2}$, be the principal subgrid of H_{t_i} obtained from H_{t_i} by removing the two outermost layers. We choose $t \approx (7r + 10)k$ so that each principal subgrid $H_{t_i}^{t-10k-7}$ contains $7k + 1$ disjoint subgrids $H_{t_i, r_1}, \dots, H_{t_i, r_{7k+1}}$ of radius r .² In this way, we get the hierarchy of nested subgrids as in Fig. 7:

$$H_s \supseteq H_{t_i} \supseteq H'_{t_i} \supseteq H_{t_i, r_j}, \quad \text{where } 1 \leq i \leq 2k + 1, \quad 1 \leq j \leq 7k + 1.$$

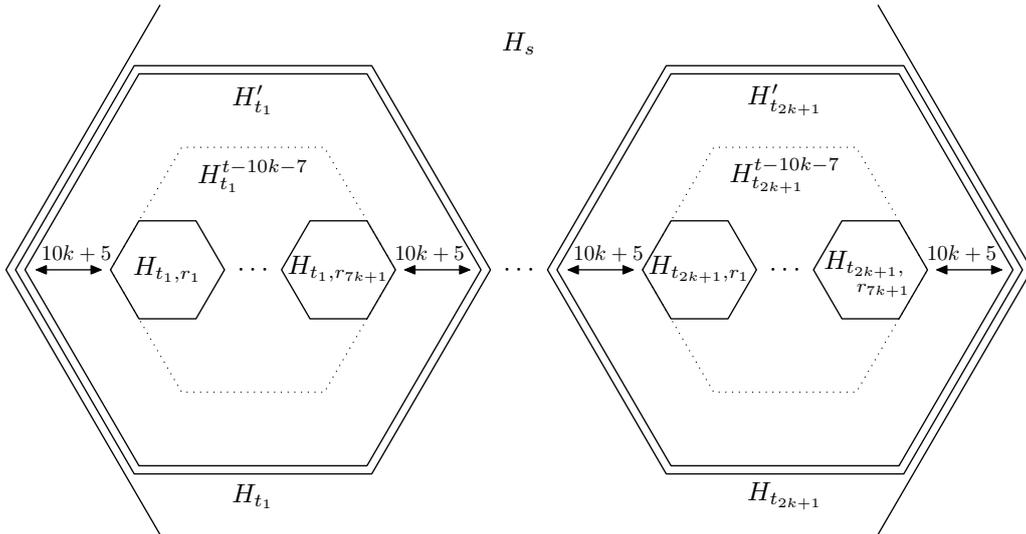


Figure 7: The hierarchy of the hexagonal grids nested in H_s .

²Actually just $s \approx \sqrt{2kt}$ and $t \approx \sqrt{7kr} + 10k$ would be sufficient.

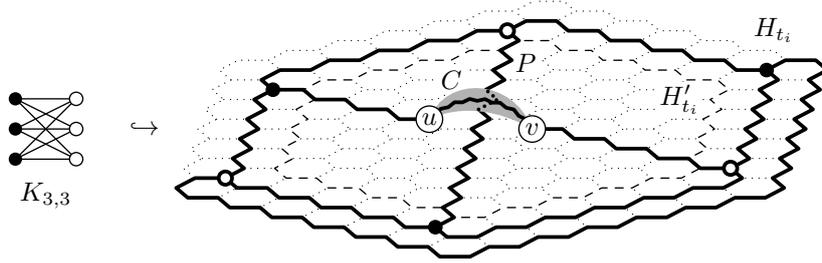


Figure 8: The component C acts as a bridge allowing two non-crossing paths across the grid. Thus we can embed $K_{3,3}$ into $h(H_{t_i})$.

We next argue using the pigeon-hole principle that for some H_{t_i, r_j} the embedding $h \upharpoonright H_{t_i, r_j}$ is a flat embedding.

Since we are assuming that G is k -contractible, we can fix a corresponding planarizing set S and consider one subgrid H_{t_i} . Let a *cell* be the h -image of a hexagon of H'_{t_i} . We call an $h(H_{t_i})$ -component *bad* if it contains an edge from S . A cell is considered *bad* if it contains an edge of S or if there is at least one bad $h(H_{t_i})$ -component attached to the cell. Since bad cells have some obstructions to planarity attached to them, we will exhibit some grid H_{t_i, r_j} such that its embedding $h(H_{t_i, r_j})$ avoids all bad cells.

To proceed, call an $h(H_{t_i})$ -component *large* if it has two vertices of attachment of C in $h(H'_{t_i})$ which do not belong to one cell of H'_{t_i} . As an example, in Fig. 6b (when $H_3 = H'_{t_i}$) the component C_3 is large, but C_1 and C_2 are not. Large $h(H_{t_i})$ -components possess the following useful properties which we prove afterwards in a series of claims.

1. If an $h(H_{t_i})$ -component is large, then we can embed $K_{3,3}$ into $h_+(H_{t_i})$. This implies that there must be some H_{t_i} having no large $h(H_{t_i})$ -component, otherwise the graph would not be k -contractible.
2. On the other hand, if a bad $h(H_{t_i})$ -component is not large, it can produce at most seven bad cells. This implies that $h(H_{t_i})$ must have a number of bad cells bounded by $7k$ and therefore for some j , the embedding $h(H_{t_i, r_j})$ contains no bad cells and it is a flat embedding.

Claim 2.4. *Let C be a large $h(H_{t_i})$ -component. Then we can embed $K_{3,3}$ into $h_+(H_{t_i})$ such that $K_{3,3}^- := K_{3,3} - e$ is embedded into the grid $h(H_{t_i})$.*

PROOF OF CLAIM. Instead of a tedious formal proof, we illustrate the main idea in Fig. 8. If C is large, it has two vertices u and v in $h(H'_{t_i})$ not contained in one cell. Thus there exists a path P going across the grid “between” u and v . Using C as a “bridge” from u to v , we can cross P by another path across the grid. These two paths together with the two outer layers of $h(H_{t_i})$ allow to embed $K_{3,3}$ into $h_+(H_{t_i})$ such that $K_{3,3}^-$ is embedded into $h(H_{t_i})$. \diamond

Claim 2.5. *There is some H_{t_i} such that there is no large $h(H_{t_i})$ -component.*

PROOF OF CLAIM. According to the above claim, if there exists a large $h(H_{t_i})$ -component, we can embed $K_{3,3}$ into $h_+(H_{t_i})$. Since $G \circ S$ is a planar graph, this embedding of $K_{3,3}$ has to be contracted by S . To contract it, there has to be an edge $e \in S$ incident with $h(K_{3,3}^-)$, otherwise $G \circ S$ still contains an embedding of $K_{3,3}$. Therefore e is incident with $h(H_{t_i})$. We know that $|S| \leq k$ and each edge in S is incident with at most two grids $h(H_{t_i})$. Since we have $2k + 1$ disjoint grids, there is some H_{t_i} such that no edge of S is incident with $h(H_{t_i})$.³ Therefore, there is no large $h(H_{t_i})$ -component. \diamond

Claim 2.6. *For H_{t_i} having no large $h(H_{t_i})$ -component, $h(H'_{t_i})$ contains at most $7k$ bad cells.*

³By a more refined analysis, one can show that only $k + 1$ disjoint grids suffice. The reason is that if an edge is incident to two grids, it contracts neither of the embeddings $h(K_{3,3})$.

PROOF OF CLAIM. Let $e \in S$. Each $h(H_{t_i})$ -component C is not large, so it is attached to at most seven cells of $h(H'_{t_i})$. Therefore, if $e \in C$, we get at most seven bad cells. If e belongs to a cell directly, we get two bad cells. Since $|S| \leq k$, we get at most $7k$ bad cells. \diamond

By the pigeon-hole principle, there exists one of $h(H_{t_i, r_1}), \dots, h(H_{t_i, r_{7k+1}})$ containing no bad cells, and we denote it by H_{t_i, r_j} .

Claim 2.7. For H_{t_i} having no large $h(H_{t_i})$ -component and $h(H_{t_i, r_j})$ having no bad cells, the embedding $h \upharpoonright H_{t_i, r_j}$ is a flat embedding.

PROOF OF CLAIM. Since $h(H_{t_i, r_j})$ contains no bad cells, clearly it contains no edges of S . Let C be a proper $h(H_{t_i, r_j})$ -component, it remains to show that $C \cap S = \emptyset$. Notice that all above claims involve $h(H_{t_i})$ -components, so we need to relate C to them. If C is an edge, let $C' = C$. If C contains an edge with one endpoint in $h(H_{t_i, r_j})$ and another endpoint in $h(H_{t_i}) \setminus h(H_{t_i, r_j})$, let C' be this edge. Otherwise let C' be a component of $C \setminus h(H_{t_i})$ together with the edges connecting C' to $h(H_{t_i})$ and its incident vertices, and assume that C' is attached to an inner vertex u of $h(H_{t_i, r_j})$; it always exists. Observe that C' is a $h(H_{t_i})$ -component.

Since $h(H_{t_i, r_j})$ contains no bad cells, $C' \cap S = \emptyset$. Also, C' is not attached to any vertex of $h(H'_{t_i} \setminus H_{t_i, r_j})$, otherwise it would be a large component. It remains to show that C' is not attached to any vertex of $h(H_{t_i} \setminus H'_{t_i})$ as well. This concludes the proof since $C \cap h(H_{t_i} \setminus H_{t_i, r_j}) = \emptyset$, so $C = C'$ and $C \cap S = \emptyset$.

Suppose that C' is attached to $v \in h(H_{t_i} \setminus H'_{t_i})$. By our assumption, u and v are at least $10k + 5$ layers of the grid $h(H_{t_i})$ apart. By the pigeon-hole principle, there exist 5 consecutive layers $L = h(H_{t_i}^q \setminus H_{t_i}^{q-5})$ of the grid $h(H'_{t_i} \setminus H_{t_i, r_j})$ such that no edge of S is incident with them. Using L , we can embed $K_{3,3}$ into $h_+(H_{t_i}) \circ S$; see Fig. 9. We embed $K_{3,3} - e$ into the middle three layers of L . The remaining edge e is embedded in the outer/inner layers of L , together with a path in $h(H_{t_i}) \cup C' \setminus L$, using a path from u to v through C' as a bridge. Since edges of S are not incident with L , they are not incident with the embedding of $K_{3,3} - e$, so $K_{3,3}$ remains in $G \circ S$, contradicting that S is a planarizing set. \diamond

Following the above claims, we get for $H_r = H_{t_i, r_j}$ a flat embedding $h \upharpoonright H_r$, concluding the proof. \square

The next lemma shows that a small part of $h_+(H_r)$, where H_r is the hexagonal grid which we got from Lemma 2.3, is never contracted by a minimal set S . Let the *core* K of $h_+(H_r)$ denote the h -image of the central principal cycle $h(C^1)$ together with the $h(H_r)$ -components attached only to $h(C^1)$.

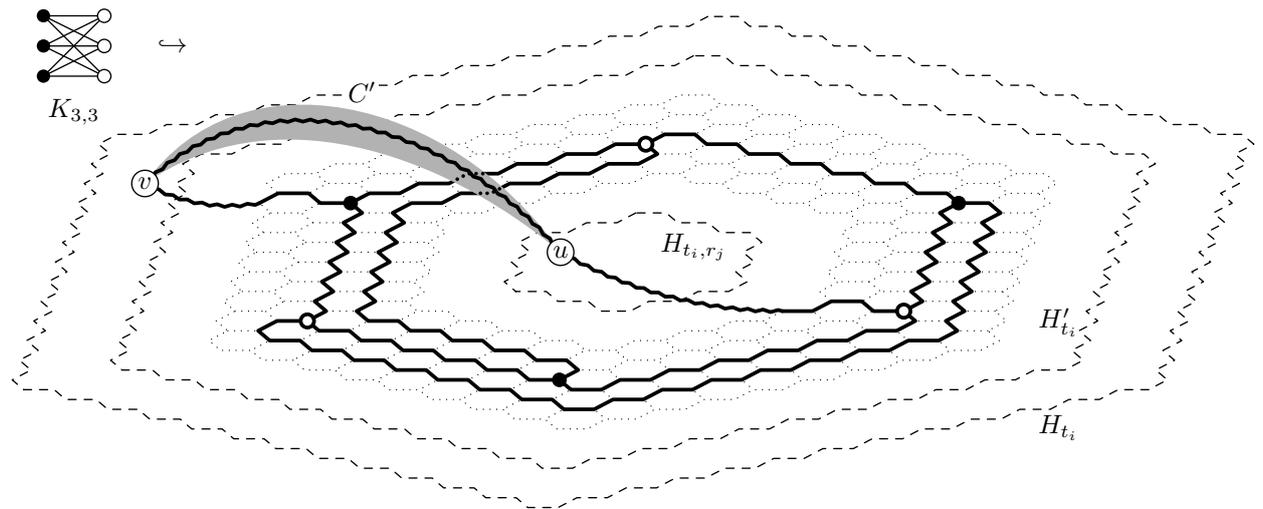


Figure 9: When 5 consecutive layers of $h(H_{t_i} \setminus H_{t_i, r_j})$ (depicted with dots) avoid the edges of S , we can use the component C' to embed $K_{3,3}$.

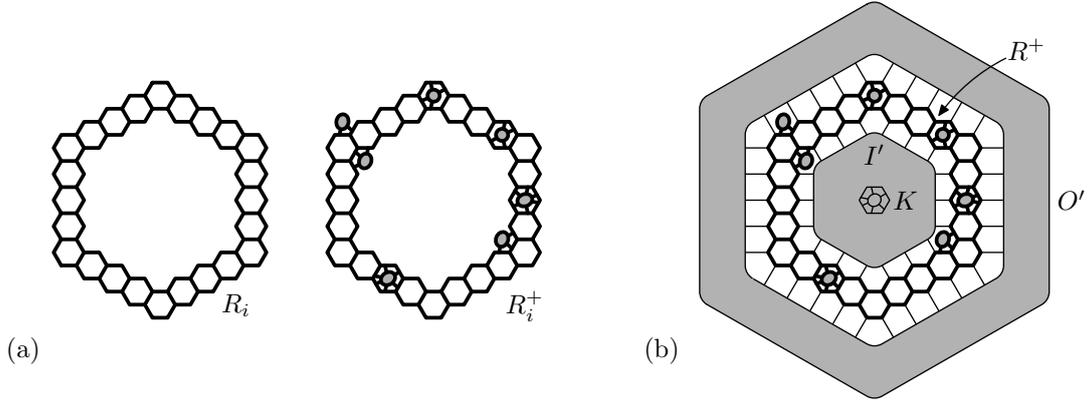


Figure 10: (a) An example of a ring R_i and an extended ring R_i^+ . (b) The extended ring R^+ (in bold) splits $G' \setminus R$ into an outer part O' and an inner part I' . The inner part is connected and contains H_r^{2i-1} and K .

Lemma 2.8. *Let G be a k -contractible graph and let S be a minimal planarizing set of G . Let $h : H_r \hookrightarrow G$, where $r \geq 4k + 3$, be a flat embedding and let K be the core of $h_+(H_r)$. Then the edges of G incident with the vertices of K do not belong to S .*

PROOF. We define a *ring* R_i as $h(H_r^{2i+1} \setminus H_r^{2i-1})$, i.e., it is the h -image of the two consecutive principal cycles of H_r together with the edges between them. An *extended ring* R_i^+ is the union of R_i with all the $h(H_r)$ -components having all vertices of attachment in R_i . See Fig. 10a. Consider $2k + 1$ disjoint extended rings R_1^+, \dots, R_{2k+1}^+ and notice that for some i , no edge of S is incident with a vertex of R_i^+ . Denote $R := R_i$, $R^+ := R_i^+$ and the principal cycles of this ring as $C_{\text{outer}} := h(C^{2i+1})$ and $C_{\text{inner}} := h(C^{2i})$.

The graph $G' := G \circ S$ is planar with a planar embedding Δ' as follows; see Fig. 10b. The ring R itself is a subdivision of a 3-connected graph, and therefore it has a unique embedding up to the choice of an outer face. We choose a drawing having C_{outer} as the outer face of $\Delta' \upharpoonright R$. For the extended ring R^+ , the embedding of the attached components is not unique, but they are attached somehow to R .

Now, $G' \setminus R^+$ is split by R^+ into two parts: the *inner part* I' lying inside the face bounded by C_{inner} and the *outer part* O' inside the outer face bounded by C_{outer} . Moreover, we can assume for Δ' that I' is connected, containing $h(H_r^{2i-1}) \circ S$. The reason is that a connected component of I' , not containing the grid, is a separate component of G' , so we choose Δ' such that it is drawn into the outer face.

Since contraction does not change connectivity in G and all contractions avoid R^+ , the subgraph $G \setminus R^+$ is separated into parts I (containing $h(H_r^{2i-1})$ and especially K) and O (containing the rest) such that $I \circ S = I'$ and $O \circ S = O'$ with no edges between I and O . We show next that the minimality of S forces that $I \cap S = \emptyset$.

We take the embedding Δ' , remove $\Delta' \upharpoonright I'$ and replace it with some embedding of I . Since $h(H_r)$ is a flat embedding, the graph $h_+(H_r)$ is planar, so in particular the subgraph induced by $R^+ \cup I$ is planar. We consider one of its planar embeddings Δ having I embedded into the inner face of R^+ . Since $\Delta \upharpoonright I$ has the same orientation of edges to R as $\Delta' \upharpoonright I'$, it is possible to replace the embedding of I' in Δ' by $\Delta \upharpoonright I$. This means that it is not necessary to contract the edges of I and therefore $I \cap S = \emptyset$. The statement follows since K and its incident edges belong to I . \square

We note that the algorithm just contracts K and not the whole I since we do not know which extended ring R_i^+ avoids edges of S .

Recall that F is the set of forbidden edges to contract. If the graph G is k -contractible by a planarizing set S such that $S \cap F = \emptyset$, we say that G is (k, F) -contractible. As we proceed with our algorithm, we use Lemma 2.8 to modify the input G and F to a smaller graph G' and an extended set F' as follows. The core K is contracted into a single vertex v , so $G' := G \circ K$. Let E_v be the set of edges incident with v . We add them to F , so $F' := F \cup E_v$. Figure 11 depicts this modification.

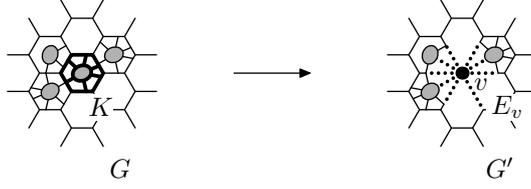


Figure 11: The graph is modified by contraction of the core K and adding its incident edges E_v to F .

Lemma 2.9. *The graph G is (k, F) -contractible if and only if the graph G' is (k, F') -contractible.*

PROOF. According to Lemma 2.8, a minimal planarizing set S for G avoiding F does not contain any edges of K and also the edges of G from which F' arises in G' . Therefore, it is also a planarizing set of G' avoiding F' .

On the other hand, assume that G' has a planarizing set S disjoint from F' of size at most k . We want to show that S is also a planarizing set for G . We consider an embedding of $G' \circ S$ and we replace the vertex created by contracting the core K by an embedding of K in a manner completely analogous to the one described in the proof of Lemma 2.8. We get a planar embedding of $G \circ S$. \square

Phase II. If the graph G is k -contractible by a planarizing set S such that $S \cap F = \emptyset$ and $P(S, G)$ is satisfied, we say that G is (k, F) -contractible with respect to P . As we process our algorithm, we use Lemma 2.8 to modify G . We show next that when the tree-width of G is small, we can solve (k, F) -contractibility with respect to P using Courcelle's theorem [13]. To this effect all we need to show is that it is possible to express (k, F) -contractibility in the monadic second-order logic (MSOL).

Lemma 2.10. *For a fixed graph H , there exists an MSOL formula $\mu_H(S, G)$ which is satisfied if and only if $G' := G \circ S$ contains H as a minor.*

PROOF. We modify a well-known formula $\tilde{\mu}_H(G)$ for testing whether H is a minor of G . For $|H| = \ell$, the formula $\tilde{\mu}_H(G)$ tests whether there exist disjoint sets of vertices V_1, \dots, V_ℓ (representing the sets of vertices contracted to vertices of H) such that

- for every $v_i v_j \in E(H)$ there exists an edge between V_i and V_j in G , and
- each set V_i is connected in G .

Let $S = \{e_1, \dots, e_k\}$ and $e_j = x_j y_j$. To test whether H is minor in G' , we require for every $j \in \{1, \dots, k\}$ that each V_i either contains both endpoints of e_j , or none of them. Formally,

$$\mu_H(S, G) = \tilde{\mu}_H(G) \wedge \bigwedge_{\substack{1 \leq i \leq \ell \\ 1 \leq j \leq k}} (x_j \in V_i \iff y_j \in V_i). \quad \square$$

Lemma 2.11. *There exists a formula $\varphi_k(F, G)$ which is satisfiable if and only if G is (k, F) -contractible with respect to the MSOL formula P .*

PROOF. This formula is defined as follows:

$$\varphi_k(F, G) := \exists S \subseteq E(G) : |S| \leq k \wedge (S \cap F = \emptyset) \wedge P(S, G) \wedge \neg \mu_{K_5}(S, G) \wedge \neg \mu_{K_{3,3}}(S, G). \quad \square$$

Putting all the Pieces Together. We finish this section with a proof of the announced Theorem 1.1, stating that P -RESTRICTEDCONTRACT for an inclusion-closed and inert contraction-closed MSOL formula $P(S, G)$ is solvable in time $\mathcal{O}(n^2 \cdot f(k))$ for some function f .

Algorithm 1 *P*-RESTRICTEDCONTRACT

Require: A graph G and an inclusion-closed and inert contraction-closed formula $P(S, G)$.

Ensure: A planarizing set S of size at most k satisfying $P(S, G)$ if it exists.

- 1: Initialize the set of forbidden edges $F := \emptyset$.
 - 2: Depending on k , choose suitable $s \geq 1$ and $t \geq 1$ for Theorem 2.1.
 - 3: **while** the tree-width of G is larger than t **do**
 - 4: Find an embedding $h : H_s \hookrightarrow G$ using Theorem 2.1.
 - 5: Find a subgrid H_r such that $h \upharpoonright H_r$ is a flat embedding as described in Lemma 2.3.
 - 6: Modify the graph as $G := G \circ K$ and $F := F \cup E_v$.
 - 7: **end while**
 - 8: **return** a planarizing set S satisfying $\varphi_k(F, G)$ using Theorem 2.2 if it exists.
-

PROOF (THEOREM 1.1). See Algorithm 1 for a pseudocode; Phase I corresponds to steps 3 to 7, Phase II corresponds to step 8. Depending on k , we choose a suitable value for s so we can apply Lemmas 2.3 and 2.8. By Theorem 2.1, there is a corresponding value of t .

We repeat Phase I till the tree-width of G becomes at most t . Every iteration of Phase I first finds embedding h of a large hexagonal grid H_s , by Theorem 2.1 in linear time. Using Lemma 2.3, there exists a subgrid H_r such that $h(H_r)$ is flat. Moreover, we can find such H_r in time $\mathcal{O}(k^2 n)$ by testing planarity for all $h_+(H_{t_i, r_j})$. We contract the kernel K and we modify the graph G and the set of forbidden edges F . Lemmas 2.8 and 2.9 show that this modification does not change the solvability of the problem. After each modification, we get a smaller graph G . Therefore we need to repeat this at most $\mathcal{O}(n)$ times, so the total running time of Phase I is $\mathcal{O}(n^2 \cdot p(k))$ for some function p .

Let G denote the original graph and let G' denote the modified graph and let F' denote the set of forbidden edges created by Phase I. Phase II uses Theorem 2.2 to solve the MSOL formula $\varphi_k(F', G')$ in time $\mathcal{O}(n \cdot q(k))$. By Lemmas 2.3, 2.8 and 2.9, the modified graph G' is (k, F') contractible if and only if the original graph is k -contractible. It remains to show that none of the modifications changes the satisfiability of $P(S, G)$. Since $P(S, G)$ is inclusion-closed, we can concentrate on inclusion-minimal planarizing sets in G and G' . Further by Lemma 2.8, each modification contracts some inert edges which are not incident with any inclusion-minimal planarizing set S . Since $P(S, G)$ is inert contraction-closed, none of these modifications changes the solvability of $P(S, G)$. So testing $\varphi_k(F', G')$ correctly tests whether G is k -contractible with respect to $P(S, G)$.

The overall complexity of the algorithm is $\mathcal{O}(n^2 \cdot f(k))$ for some function f . □

3. ℓ -subgraph Contractibility

We first establish Corollary 1.3 which states that for a fixed ℓ , testing ℓ -subgraph contractibility can be done in time $\mathcal{O}(n^2 \cdot f'_\ell(k))$ for some function f'_ℓ . It follows from Theorem 1.1 and the fact that ℓ -subgraph contractibility is expressible using MSOL.

PROOF (COROLLARY 1.3). We just describe in words how to construct the MSOL formula P and we check that P is inclusion-closed and inert contraction-closed. The length of the formula may depend on k and ℓ . The formula P tests whether there exists a decomposition of the edges in S into pairwise disjoint sets E_1, \dots, E_k satisfying the following properties. Let V_1, \dots, V_k denote the corresponding sets of vertices incident with E_1, \dots, E_k , respectively. The formula P is satisfied if and only if $|V_i| \leq \ell$ for each i and the sets V_1, \dots, V_k are pairwise disjoint. This solves ℓ -subgraph contractibility and is an inclusion-closed MSOL formula.

It remains to show that P is inert contraction-closed. Assume that $P(S, G)$ is satisfiable and let S be an inclusion-minimal planarizing set satisfying $P(S, G)$. For every inert set B , by definition, no edge of B is incident with an edge of S . Therefore S is a planarizing set of $G \circ B$ consisting of ℓ -subgraphs, and

$P(S, G \circ B)$ is satisfiable. For the other implication, if S is an inclusion-minimal planarizing set satisfying $P(S, G \circ B)$, then it also satisfies $P(S, G)$. The reason is that B is inert, so no edge of S is incident with an edge of B . Therefore, S is a planarizing set of G consisting of ℓ -subgraphs. \square

Matching Contractibility. In the rest of this section, we establish NP-completeness of ℓ -SUBCONTRACT. To do so, we first introduce a new problem called *matching contractibility*. The graph G is *F-matching contractible* with respect to a set of edges F if there exists a planarizing set S which forms a matching in G and $S \cap F = \emptyset$.

Problem:	MATCHINGCONTRACT
Input:	An undirected graph G and a set of forbidden edges $F \subseteq E$.
Output:	Is G an F -matching contractible graph?

First, we show that ℓ -subgraph contractibility can solve matching contractibility.

Lemma 3.1. *Matching contractibility is polynomial-time reducible to ℓ -subgraph contractibility, for any fixed ℓ .*

PROOF. For an input G and F , we produce a graph G' which is ℓ -subgraph contractible if and only if G is F -matching contractible. We replace the edges of G by paths:

- if $e \in F$, then we replace it by a path of length ℓ , and
- if $e \notin F$, then we replace it by a path of length $\ell - 1$.

Also, we put $k = |E(G')|$ so only ℓ -subgraphs restrict a planarizing set.

If a planarizing set S is a matching in G avoiding F , then we can contract the corresponding paths in G' by ℓ -subgraphs. On the other hand, let S' be a planarizing set of G' consisting of ℓ -subgraphs. First, we ignore each ℓ -subgraph which does not contract one of the paths of length $\ell - 1$, since its contraction preserves the topological structure of the graph. If a path in G' corresponding to $e \in E(G)$ is contracted, it has to be contracted by a single ℓ -subgraph. In such a case, $e \notin F$, otherwise the path is too long. Also, the contracted paths have to form a matching since the ℓ -subgraphs cannot share the end-vertices belonging to G . So the planarizing set S' of G' gives a planarizing set S of G which is a matching and which avoids F . \square

Overview of the Reduction. To show NP-hardness of MATCHINGCONTRACT, we present a reduction from CLAUSE-LINKED PLANAR 3-SAT. An instance I of CLAUSE-LINKED PLANAR 3-SAT is a Boolean formula in CNF such that each variable occurs in exactly three clauses, once negated and twice positive, each clause contains two or three literals and the incidence graph of I is planar. Fellows et al. [18] show that this problem is NP-complete.

Given a formula I , we construct a graph G_I with a set F_I of forbidden edges such that G_I is F_I -matching contractible if and only if I is satisfiable. The construction has a variable gadget G_x for each variable x , and a clause gadget H_c for each clause c . All variable gadgets G_x are isomorphic and we have two types of clause gadgets H_c , depending on the size of c . These gadgets consist of several copies of the graph K_5 with most of the edges in F_I . In Fig. 12, the edges not contained in F_I are represented by bold lines. Each variable gadget contains three pendant edges that are identified with certain edges of the clause gadgets, thus connecting the variable and clause gadgets.

Variable Gadget. Let x be a variable which occurs positively in clauses c_1 and c_2 , and negatively in a clause c_3 . The corresponding *variable gadget* G_x is depicted in Fig. 12a. It consists of four copies of K_5 , each having all but two edges in F_I . Three of the copies of K_5 have pendant edges attached, denoted by

$$e(x, c_i) = v(x, c_i)w(c_i), \quad i \in \{1, 2, 3\};$$

refer to Fig. 12a. These edges also belong to the clause gadgets H_{c_1} , H_{c_2} and H_{c_3} . All other vertices and edges are private to the variable gadget G_x .

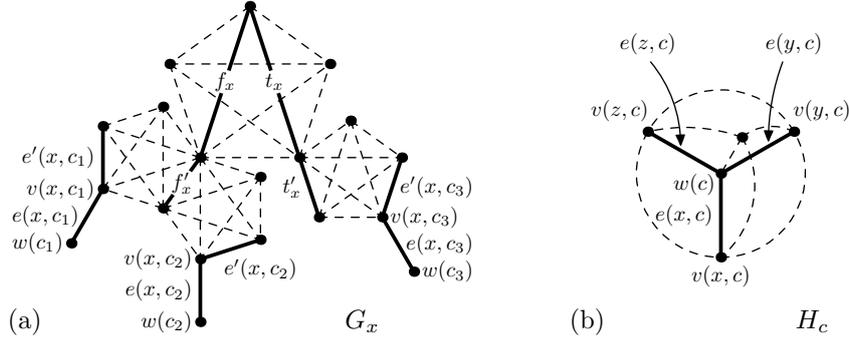


Figure 12: The bold edges can be contracted and the dashed edges are forbidden edges from F_I . (a) The variable gadget G_x where the three outgoing edges are shared with clause gadgets H_{c_1} , H_{c_2} and H_{c_3} . (b) The clause gadget H_c . The edge $e(z, c)$ may also be in F_I if the clause contains only two variables x and y . The two or three contractible edges are shared with variable gadgets G_x , G_y and possibly G_z .

The main idea behind the variable gadget is that exactly one of the edges t_x and f_x is contracted. This encodes the assignment of the variable x as follows: the edge t_x is contracted for true and f_x for false. The edges $e(x, c_1)$ and $e(x, c_2)$ shared with the clause gadgets H_{c_1} and H_{c_2} , can be contracted only if t_x is contracted, and $e(x, c_3)$ shared with H_{c_3} can be contracted if and only if f_x is contracted.

Clause Gadget. Let c be a clause containing variables x , y and possibly z . The *clause gadget* H_c is the graph K_5 with all but 2 or 3 edges in F_I . The edges that are not forbidden to contract share a common vertex $w(c)$ and they are the edges $e(x, c)$, $e(y, c)$ and possibly $e(z, c)$ shared with the variable gadgets G_x , G_y and possibly G_z ; see Fig. 12b.

To make the clause gadget planar, we need to contract exactly one of the edges $e(x, c)$, $e(y, c)$ and possibly $e(z, c)$. This is possible only if the clause is satisfied by the corresponding variable evaluated as true in this clause.

Lemma 3.2. *The graph G_I is F_I -matching contractible if and only if I is satisfiable.*

PROOF. \implies : Suppose first that G_I is F_I -matching contractible, and let $S \subseteq E(G_I) \setminus F_I$ be a matching planarizing set. Using S , we construct a satisfying assignment of I . Consider a variable x . In G_x , each copy of K_5 needs to have at least one edge contracted by S .

Exactly one of t_x and f_x is in S . If $t_x \in S$, then t'_x cannot be in S (note that S is a matching), hence $e'(x, c_3) \in S$, and $e(x, c_3)$ cannot be in S . On the other hand, if $t_x \notin S$, necessarily $f_x \in S$, and by a similar sequence of arguments, none of $e(x, c_1)$ and $e(x, c_2)$ is in S .

We define a truth assignment for the variables of I so that x is true if and only if $t_x \in S$. It follows that if x appears as a false literal in a clause c , then the edge $e(x, c)$ is not in S . Since S contains exactly one edge of H_c , in each clause gadget at least one literal must be evaluated to true. Thus I is satisfiable.

\impliedby : Suppose that I is satisfiable and fix a satisfying truth assignment ϕ . We set

$$S = \{t_x, f'_x, e(x, c_1), e(x, c_2), e'(x, c_3) \mid \phi(x) = \text{true}\} \cup \{f_x, t'_x, e'(x, c_1), e'(x, c_2), e(x, c_3) \mid \phi(x) = \text{false}\}.$$

The edges of S contained in variable gadgets form a matching. Each clause gadget contains at least one edge of S . But if a clause, say c , contains more than one literal evaluated as true, then its clause gadget H_c contains in S more edges with the common vertex $w(c)$. In such a case, we perform a *pruning operation* on H_c , i.e., remove all edges from $S \cap E(H_c)$ but one. The resulting set S' is a matching such that each K_5 in G_I contains exactly one edge of S' .

It only remains to argue that this set S' is a planarizing set. The graph $G' = G \circ S'$ consists of copies of K_4 glued together by vertices or edges. Each copy is attached to other copies by at most three vertices. Since K_4 itself has a non-crossing drawing in the plane such that three of its vertices lie on the boundary

of the outer face, and these vertices can be chosen arbitrarily as well as their cyclic order on the outer face, the drawings of the variable and clause gadgets can be combined together along a planar drawing of the incidence graph of I . Therefore, G' is planar. \square

This shows that ℓ -SUBCONTRACT is NP-complete for every $\ell \geq 2$:

PROOF (PROPOSITION 1.2). Clearly, ℓ -SUBCONTRACT belongs to NP. By Lemma 3.2 and [18], the problem MATCHINGCONTRACT is NP-hard. Lemma 3.1 implies that ℓ -SUBCONTRACT is NP-hard. \square

We note that the problem ℓ -SUBCONTRACT remains NP-complete when generalized to surfaces of a fixed genus g (instead of planar graphs). Consider a graph H_g such that for every embedding of H_g into the surface, each face is homeomorphic to the disk. We modify our reduction by taking $G_I \cup H_g$ as the graph and by adding all the edges of H_g into F . For each surface, there exists such a graph H_g (see triangulated surfaces in Mohar and Thomassen [19]).

4. Simplifying Grohe's Approach

Grohe [11] gives an FPT algorithm for computing the crossing number k of a graph, in time $\mathcal{O}(n^2 f(k))$ for some function f . Our FPT algorithm is based on his approach. On the other hand, we can simplify his argument in a similar manner as in the proof of Lemma 2.3. We describe this simplification here.

Grohe uses Thomassen's Theorem [20] which states the following:

Theorem 4.1 (Thomassen [20]). *Let G be a graph of genus at most k . For every $r \geq 1$, there is $s \geq 1$ such that for every topological embedding $h : H_s \rightarrow G$, there exists a subgrid $H_r \subseteq H_s$ such that the restriction $h \upharpoonright H_r$ of h is flat.*

This result can be used since for graphs the genus is upper bounded by the crossing number, so it works as Lemma 2.3 in our FPT algorithm. With our simplification, we can completely avoid the notion of genus which is independent of the notion of crossing number. Therefore, we do not need to consider the more complicated theory of graphs on surfaces. Also, our proof is quite short and elementary.

Consider a plane drawing D of a graph G . The *facial distance* $d(x, y)$ of vertices x, y in G is the minimal number of intersections of D and a simple curve C connecting x and y , where C avoids the vertices of G . Let $\text{cr}(G)$ denote the crossing number of a graph G . We use Riskin's Theorem [21]:

Theorem 4.2 (Riskin [21]). *If G is a 3-connected cubic planar graph, then*

$$\text{cr}(G + xy) = d(x, y),$$

where $G + xy$ is the graph G with the added edge xy .

Lemma 4.3. *Let G be a graph with $\text{cr}(G) \leq k$. For every $r \geq 1$, there is $s \geq 1$ such that for every topological embedding $h : H_s \rightarrow G$, there exists a subgrid $H_r \subseteq H_s$ such that the restriction $h \upharpoonright H_r$ of h is flat.*

PROOF. Similarly as in Lemma 2.3, we define a hierarchy of grids as in Fig. 13. We choose $s \approx (k + r) \cdot k$ so that H_s contains $k + 1$ disjoint subgrids $H_{t_1}, \dots, H_{t_{k+1}}$ of radius $k + r$. Let H'_{t_i} denote the principal subgrid $H_{t_i}^r$ of H_{t_i} . In this way, we get the hierarchy:

$$H_s \supseteq H_{t_i} \supseteq H'_{t_i}, \quad \text{where } 1 \leq i \leq k + 1.$$

We want to show using the pigeon-hole principle that for some H'_{t_i} the embedding $h \upharpoonright H'_{t_i}$ is a flat embedding.

Claim 4.4. *Let C be an $h(H_s)$ -component which has a vertex of attachment in $h(H'_{t_i}) \setminus h(C_{t_i}^r)$, where $C_{t_i}^r$ is the r -th principal cycle of H_{t_i} . Then C has no attachment vertices in $h(H_s) \setminus h(H_{t_i})$.*

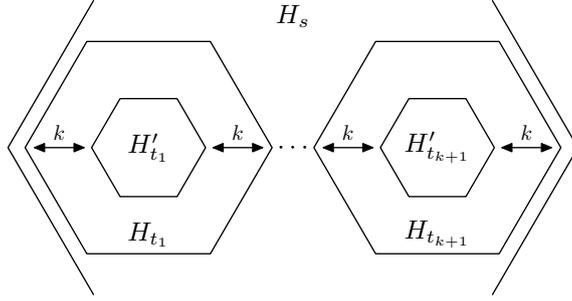


Figure 13: The hierarchy of the hexagonal grids nested in H_s .

PROOF OF CLAIM. For contradiction, let x be an inner vertex of $h(H'_{t_i})$ which is a vertex of attachment of C , and let y be a vertex of attachment of C in $h(H_s) \setminus h(H_{t_i})$. Then there exists a path P from x to y with no internal vertices in $h(H_s)$. Consider $h(H_s)$ together with P and apply Riskin's Theorem 4.2. (We note that subdividing a graph preserves the crossing number.) Since the face distance $d(x, y)$ is at least $k + 1$, we get that $\text{cr}(G) > k$ which is a contradiction. \diamond

Let G_i be the graphs $h(H_{t_i})$ together with $h(H_{t_i})$ -components with vertices of attachment in $h(H'_{t_i}) \setminus h(C_{t_i}^r)$ (which are also $h(H_s)$ -components, by the previous claim).

Claim 4.5. *Let G_i be the graph defined above. If G_i is planar, then the embedding $h \upharpoonright H'_{t_i}$ is flat.*

PROOF OF CLAIM. Let C be a proper $h(H'_{t_i})$ -component. Exactly as in the proof of Claim 2.7, we construct from C a proper $h(H_{t_i})$ -component C' . Since C' has a vertex of attachment in $h(H'_{t_i}) \setminus h(C_{t_i}^r)$, it belongs to G_i .

The graph $h(H_{t_i})$ is a subdivision of a 3-connected planar graph, so it has a unique embedding into the plane having $h(C_{t_i}^{r+k})$ as the outerface. Since G_i is planar, the component C' has to be embedded into a face bounded by a cell of $h(H_{t_i})$ (which is the h -image of a hexagon of H'_{t_i}). Therefore, C' has no vertices of attachment in $h(H_{t_i}) \setminus h(H'_{t_i})$ and $C = C'$. Thus, $h_+(H'_{t_i})$ is constructed from G_i by removing $h(H_{t_i}) \setminus h(H'_{t_i})$, so it is planar. \diamond

By Claim 4.4, graphs G_1, \dots, G_{k+1} are pairwise disjoint. Since $\text{cr}(G) \leq k$, by the pigeon-hole principle, some G_i is planar. By Claim 4.5, the embedding $h \upharpoonright H'_{t_i}$ is flat. \square

5. Open Problems

We conclude this paper with several open problems.

Problem 1. *Let $P(S, G)$ be an inclusion-closed and inert contraction-closed MSOL formula. Can the problem P -RESTRICTEDCONTRACT be solved in time $\mathcal{O}(n \cdot \tilde{f}(k))$?*

Problem 2. *Consider the generalization of P -RESTRICTEDCONTRACT which asks whether there exists a set $S \subseteq E(G)$ such that $|S| \leq k$ and $G \circ S$ is a graph of genus at most g . Is this problem fixed-parameter tractable with the parameter k ?*

In generalizing our approach, the main difficulty lies in Lemma 2.3.

Asano and Hirata [6] proved that for an MSOL formula $P(S, G)$ which is always satisfied P -RESTRICTEDCONTRACT is NP-complete. In Section 3, we show this for one particular MSOL formula $P(S, G)$ which test whether S consists of ℓ -subgraphs. The very natural question is for which formulas $P(S, G)$ it is NP-complete. Clearly, it is not for every formula $P(S, G)$. For instance, if the formula $P(S, G)$ cannot be satisfied at all, the problem P -RESTRICTEDCONTRACT can be solved by outputting “no” and clearly belongs to P.

Problem 3. For which MSOL formulas $P(S, G)$ is the problem P -RESTRICTEDCONTRACT NP-complete?

Theorem 1.1 shows that for every inclusion-closed and inert contraction-closed formula $P(S, G)$, the problem P -RESTRICTEDCONTRACT can be solved in FPT time with respect to the parameter k . Can this be strengthened?

Problem 4. For which MSOL formulas $P(S, G)$ is the problem P -RESTRICTEDCONTRACT solvable in FPT time with respect to the parameter k ?

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